Last time we saw:

$$\mathcal{L}_A = -\frac{1}{4} g_{\alpha\beta} F^{\alpha}{}_{\mu\nu} F^{\beta\mu\nu}$$

where $g_{\alpha\beta}$ satisfies

$$g_{\alpha\beta} C^{\beta}{}_{\gamma\delta} = - g_{\gamma\beta} C^{\beta}{}_{\alpha\delta} \qquad (*)$$

and $g_{\alpha\beta}$ positive-definite

Theorem 1:
$\Rightarrow$ Lie algebra is direct sum of commuting compact simple and $U(1)$ subalgebras $(**)$

Such Lie algebras have Hermitian generators t_α

Set: $g_{\alpha\beta} = \mathrm{Tr}\{t_\alpha t_\beta\}$

$\rightarrow$ is positive-definite since $g_{\alpha\beta} u^\alpha u^\beta = \mathrm{Tr}\{(u^\alpha t_\alpha)^2\} \geq 0$
$\forall$ $u^\alpha \in \mathbb{R}$ and vanishes only if $u^\alpha t_\alpha = 0$
(recall t_α are hermitian: $t_\alpha^T = t_\alpha^*$)

Also condition $(*)$ is satisfied:

$$i C^{\gamma}{}_{\alpha\beta} T\{t_\gamma t_\delta\} = \mathrm{Tr}\{[t_\alpha, t_\beta] t_\delta\} = \mathrm{Tr}\{t_\delta t_\alpha t_\beta - t_\beta t_\alpha t_\delta\}$$

$\rightarrow$ anti-symmetric in β and δ.

Theorem 1 $\Rightarrow$ $(**)$

In dimension 3 an example of this is the $su(2)$ Lie algebra: $[t_\alpha, t_\beta] = i\varepsilon_{\alpha\beta\gamma} t_\gamma$

Theorem 2:

(*) implies $g_{ma,nb} = g_m^{-2} \delta_{mn} \delta_{ab}$

where the notation implies that the g_m real, Lie algebra $\mathfrak{g}$ is a direct sum

$$\mathfrak{g} = \bigoplus_m \mathfrak{g}_m$$

simple or $U(1)$

with $\{t_{ma}\}$ being generators of $\mathfrak{g}_m$

$\longrightarrow$ eliminate g_m^{-2} by rescaling

$$A^\mu_{ma} \longrightarrow \tilde{A}^\mu_{ma} \equiv g_m^{-1} A^\mu_{ma} ,$$

$$t_{ma} \longrightarrow \tilde{t}_{ma} = g_m t_{ma} ,$$

$$C^{(m)}_{cab} \longrightarrow \tilde{C}^{(m)}_{cab} = g_m C^{(m)}_{cab} .$$

$\longrightarrow \quad g_{\alpha\beta} = \delta_{\alpha\beta}$

What is then the meaning of g_m ?

It is the "coupling constant" of the gauge th.!

§1.3 Field Equations and Conservation Laws

Full Lagrangian density:

$$\mathcal{L} = -\frac{1}{4} F_{\alpha\mu\nu} F_{\alpha}^{\ \mu\nu} + \underset{\substack{\uparrow \\ \text{matter Lag. density}}}{\mathcal{L}_M}(\psi, D_\mu \psi)$$

Equations of motion:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_{\alpha\nu})} = -\partial_\mu F_\alpha^{\ \mu\nu} = \frac{\partial \mathcal{L}}{\partial A_{\alpha\nu}}$$

$$= -F_\gamma^{\ \nu\mu} C_{\gamma\alpha\beta} A_{\beta\mu} - i \frac{\partial \mathcal{L}_M}{\partial D_\nu \psi} t_\alpha \psi$$

$$\longrightarrow \partial_\mu F_\alpha^{\ \mu\nu} = -\tilde{J}_\alpha^{\ \nu} \qquad (1)$$

where $\tilde{J}_\alpha^{\ \nu}$ is the current:

$$\tilde{J}_\alpha^{\ \nu} \equiv -F_\gamma^{\ \nu\mu} C_{\gamma\alpha\beta} F_{\beta\mu} - i \frac{\partial \mathcal{L}_M}{\partial D_\nu \psi} t_\alpha \psi$$

The current $\tilde{J}_\alpha^{\ \nu}$ is conserved:

$$\partial_\nu \tilde{J}_\alpha^{\ \nu} = 0 \qquad (\partial_\nu \partial_\mu F_\alpha^{\ \mu\nu} = 0)$$

we can also rewrite eq. (1) in terms of covariant derivatives:

$$D_\nu F_\alpha^{\ \mu\nu} \equiv \partial_\lambda F_\alpha^{\ \mu\nu} - i (t_\beta^{A})_{\alpha\gamma} A_{\beta\lambda} F_\gamma^{\ \mu\nu}$$

$$= \partial_\lambda F_\alpha^{\ \mu\nu} - C_{\alpha\gamma\beta} A_{\beta\lambda} F_\gamma^{\ \mu\nu}$$

$\longrightarrow$ eq. (1) becomes:

$$D_\mu F_\alpha^{\ \mu\nu} = -J_\alpha^{\ \nu}, \quad \text{with } J_\alpha^{\ \nu} \equiv -i \frac{\partial \mathcal{L}_M}{\partial D_\nu \psi} t_\alpha \psi$$

Using

$$[D_\nu, D_\mu] F_\alpha{}^{\rho\sigma} = -i(t^A{}_\gamma)_{\alpha\beta} F_{\gamma\nu\mu} F_\beta{}^{\rho\sigma} = -C_{\gamma\alpha\beta} F_{\gamma\nu\mu} F_\beta{}^{\rho\sigma}$$

we can see

$$D_\nu J_\alpha{}^\nu = 0$$

We can also derive

$$D_\mu F_{\alpha\nu\lambda} + D_\nu F_{\alpha\lambda\mu} + D_\lambda F_{\alpha\mu\nu} = 0$$

Analogy to GR:

$$\partial_\mu F_\alpha{}^{\mu\nu} = -J_\alpha{}^\nu \longleftrightarrow R^\nu{}_\mu - \frac{1}{2}\delta^\nu{}_\mu R = -8\pi G T^\nu{}_\mu$$

$$\partial_\nu T^\nu{}_\mu \neq 0$$

$$D_\mu F_\alpha{}^{\mu\nu} = -J_\alpha{}^\nu \longleftrightarrow \left(R^\nu{}_\mu - \frac{1}{2}\delta^\nu{}_\mu R\right)\Big|_{\text{linear}} = -8\pi G \tau^\nu{}_\mu$$

$$\text{where } \tau^\nu{}_\mu \equiv T^\nu{}_\mu + \frac{1}{8\pi G}\left(R^\nu{}_\mu - \frac{1}{2}\delta^\nu{}_\mu R\right)\Big|_{\text{non-linear}}$$

We have

$$\partial_\nu \tau^\nu{}_\mu = 0$$

→ current of <u>total</u> energy and momentum

$$P_\mu = \int \tau^0{}_\mu \, d^3x$$

↑ carries purely gravitational term

→ gravitational field carries energy and momentum

§1.4 Quantization

$$\mathcal{L} = -\frac{1}{4} F_{\alpha\mu\nu} F_\alpha^{\ \mu\nu} + \mathcal{L}_M(\psi, D_\mu\psi)$$

with
$$F_{\alpha\mu\nu} = \partial_\mu A_{\alpha\nu} - \partial_\nu A_{\alpha\mu} + C_{\alpha\beta\gamma} A_{\beta\mu} A_{\gamma\nu},$$
$$D_\mu \psi = \partial_\mu \psi - i t_\alpha A_{\alpha\mu} \psi$$

Constraints:

(1) $$\Pi_{\alpha 0} \equiv \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\alpha^0)} = 0$$

(2) $$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_{\alpha 0})} + \frac{\partial \mathcal{L}}{\partial A_{\alpha 0}} = \partial_\mu F_\alpha^{\ \mu 0} + F_\gamma^{\ \mu 0} C_{\gamma\alpha\beta} A_{\beta\mu} + J_\alpha^0$$
$$= \partial_\kappa \Pi_\alpha^\kappa + \Pi_\gamma^\kappa C_{\gamma\alpha\beta} A_{\beta\kappa} + J_\alpha^0 = 0$$

where $\Pi^\kappa_\alpha \equiv \dfrac{\partial \mathcal{L}}{\partial(\partial_0 A_{\alpha\kappa})} = F_\alpha^{\ \kappa 0}$ with $\kappa = 1, 2, 3$.

We deal with these constraints by choosing a gauge.

$\longrightarrow$ "axial gauge": $A_{\alpha 3} = 0$ $\quad (*)$

canonical variables for gauge field: $A_{\alpha i}$, $i = 1, 2$

canonical momenta:
$$\Pi_{\alpha i} \equiv \frac{\partial \mathcal{L}}{\partial(\partial_0 A_{\alpha i})} = -F_\alpha^{\ 0i} = \partial_0 A_{\alpha 0} + C_{\alpha\beta\gamma} A_{\beta 0} A_{\gamma i}$$

$A_{\alpha 0}$ is not independent and can be solved for:
$$F_\alpha^{\ i0} = \Pi_{\alpha i}, \quad F_\alpha^{\ 30} = \partial_3 A_\alpha^{\ 0} \qquad (**)$$

→ constraint (2) becomes

$$-(\partial_3)^2 A_\alpha{}^0 = \partial_i \Pi_{\alpha i} + \Pi_{\gamma i} C_{\gamma\alpha\beta} A_{\beta i} + J_\alpha{}^0 \qquad (3)$$

→ can be solved for $A_\alpha{}^0$ (imposing boundary conditions)

canonical conjugate to matter field:

$$\pi_\ell = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi_\ell)} = \frac{\partial \mathcal{L}_M}{\partial(D_0 \psi_\ell)}$$

→ matter current becomes:

$$J^0_\alpha = -i \frac{\partial \mathcal{L}_M}{\partial(D_0\psi)_\ell} (t_\alpha)_\ell{}^m \psi_m = -i \pi_\ell (t_\alpha)_\ell{}^m \psi_m$$

→ (3) defines $A_\alpha{}^0$ at a given time as a functional of the canonical variables $\Pi_{\gamma i}$, $A_{\beta i}$, π_ℓ, and ψ_ℓ.

Hamiltonian is obtained by Legendre-transformation of $\mathcal{L}$:

$$\begin{aligned}\mathcal{H} &= \Pi_{\alpha i} \partial_0 A_{\alpha i} + \pi_\ell \partial_0 \psi_\ell - \mathcal{L} \\ &= \Pi_{\alpha i}(F_{\alpha 0 i} + \partial_i A_{\alpha 0} - C_{\alpha\beta\gamma} A_{\beta 0} A_{\gamma i}) + \pi_\ell \partial_0 \psi_\ell \\ &\quad - \tfrac{1}{2} F_{\alpha 0 i} F_{\alpha 0 i} + \tfrac{1}{2} F_{\alpha i j} F_{\alpha i j} + \tfrac{1}{2} F_{\alpha i 3} F_{\alpha i 3} \\ &\quad - \tfrac{1}{2} F_{\alpha 0 3} F_{\alpha 0 3} - \mathcal{L}_M .\end{aligned}$$

Using (*) and (**), this is

$$\begin{aligned}\mathcal{H} &= \mathcal{H}_M + \Pi_{\alpha i}(\partial_i A_{\alpha 0} - C_{\alpha\beta\gamma} A_{\beta 0} A_{\gamma i}) + \tfrac{1}{2} \Pi_{\alpha i} \Pi_{\alpha i} \\ &\quad + \tfrac{1}{4} F_{\alpha i j} F_{\alpha i j} + \tfrac{1}{2} \partial_3 A_{\alpha i} \partial_3 A_{\alpha i} - \tfrac{1}{2} \partial_3 A_{\alpha 0} \partial_3 A_{\alpha 0}\end{aligned}$$

where $\mathcal{H}_M$ is the matter Hamiltonian density:

$$\mathcal{H}_M \equiv \pi_\ell \partial_0 \psi_\ell - \mathcal{L}_M$$

→ path integral over $A_{\alpha i}$, $\Pi_{\alpha i}$, ψ_ℓ, and π_ℓ, with weighting factor $\exp(iI)$, where

$$I = \int d^4x \left[\Pi_{\alpha i} \partial_0 A_{\alpha i} + \pi_\ell \partial_0 \psi_\ell - \mathcal{H} + \varepsilon \text{ terms} \right],$$

<u>Note:</u> I is a quadratic functional of all fields!

→ path integral over Gaussian gives saddle-point.

Treating $A_{\alpha 0}$ as an independent variable gives the saddle point:

$$0 = \frac{\delta I}{\delta A_{\alpha 0}} = -\frac{\partial \mathcal{H}}{\partial A_{\alpha 0}} = J_\alpha{}^0 + \partial_i \Pi_{\alpha i} + C_{\beta\alpha\gamma} \Pi_{\beta i} A_{\gamma i} - \partial_3^2 A_{\alpha 0}$$

→ gives back constraints (2)

→ we are allowed to treat $A_{\alpha 0}$ as independent in path integral.

Stationary points of action are:

$$0 = \frac{\delta I}{\delta \pi_\ell} = \partial_0 \psi_\ell - \frac{\partial \mathcal{H}_M}{\partial \pi_\ell},$$

$$0 = \frac{\delta I}{\delta \Pi_{\alpha i}} = \partial_0 A_{\alpha i} - \Pi_{\alpha i} - \partial_i A_{\alpha 0} + C_{\alpha\beta\gamma} A_{\beta 0} A_{\gamma i}$$
$$= F_{\alpha 0 i} - \Pi_{\alpha i} .$$

Inserting back gives:

$$I = \int d^4x \left[\mathcal{L}_M + \frac{1}{2} F_{\alpha 0 i} F_{\alpha 0 i} - \frac{1}{4} F_{\alpha i j} F_{\alpha i j} - \frac{1}{2} \partial_3 A_{\alpha i} \partial_3 A_{\alpha i} + \frac{1}{2} (\partial_3 A_{\alpha 0})^2 \right]$$

$= \int d^4x \, \mathcal{L}$

which is our original Lagrangian!

→ We obtain the following path-integral formula:

$$\langle T\{\mathcal{O}_A \mathcal{O}_B \cdots\}\rangle_{\text{VACUUM}} \sim \int \left[\prod_{\ell,x} d\psi_\ell(x)\right]\left[\prod_{\alpha,\mu,x} dA_{\alpha\mu}(x)\right]$$

$$\times\, \mathcal{O}_A \mathcal{O}_B \cdots \exp\{iI + \varepsilon \text{ terms}\} \prod_{x,\alpha} \delta(A_{\alpha 3}(x))$$

→ not manifestly Lorentz-invariant!